World-to-Eye Transformation

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Given the position of the viewer’s eye, say, \((e_x, e_y, e_z)\) and the viewer’s center of interest, \((c_x, c_y, c_z)\), we wish to transform the vector from the eye to the center of interest so that the viewer is centered at the origin and looking down along the positive \(z\)-axis. This transform facilitates the future tasks of perspective viewing, clipping, backface elimination, light modelling, etc.

There are four steps necessary to accomplish the world-to-eye transformation. In order, they are as follows:

1. Translate the eye to the origin.
2. Rotate about the \(y\)-axis so that the translated centered of interest (from step 1) moves into the \(y\)-\(z\) plane.
3. Rotate about the \(x\)-axis so that the rotated center of interest (from step 2) is aligned with the \(z\)-axis.
4. Negate the \(x\) values (i.e. reflect about the \(y\)-\(z\) plane), since we normally prefer the positive \(x\)-axis on the viewer’s right.

1 Translate to the origin

To translate the eye point to the origin we multiply by the following matrix:

\[ T = \begin{bmatrix}
1 & 0 & 0 & -e_x \\
0 & 1 & 0 & -e_y \\
0 & 0 & 1 & -e_z \\
0 & 0 & 0 & 1
\end{bmatrix} \]
2 Rotate about the $y$-axis

Consider Figure 1. Here we have the vector from the eye to the center of interest as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c_x - e_x \\ e_y - e_y \\ c_z - e_z \end{bmatrix}$$

and the length of this vector being $r = \sqrt{a^2 + b^2 + c^2}$. The vector $< a, 0, c >$ is the “shadow” of the vector $< a, b, c >$ (assuming an infinitely far away light source above the $x$-$z$ plane). The length of vector $< a, 0, c >$ is $p = \sqrt{a^2 + c^2}$

Figure 1: Before rotating about $y$-axis through an angle of $-\theta$

We now wish to rotate the vector $< a, b, c >$ into the $y$-$z$ plane (I'm beginning to sound like Tatoo on Fantasy Island). To do this, we must rotate $\theta$ degrees about the $y$-axis. How do we construct such a matrix?

2.1 Determining the matrix to rotate about the $z$-axis

As an aside, consider the matrix to rotate a point about the $z$-axis. Note that rotating about the $z$-axis is similar to what we have already done in 2D when rotating a point about the origin. Also notice that such a rotation doesn’t change the $z$ coordinate.

So, we would expect the matrix to rotate $\beta$ degrees about the $z$-axis to
be,

$$R_z(\beta) = \begin{bmatrix} \cos \beta & -\sin \beta & 0 & 0 \\ \sin \beta & \cos \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To verify that this matrix is indeed correct, we will rotate the point 
(1, 0, 0, 1) by 90° then back again by rotating by −90° to obtain the original point.

$$\begin{bmatrix} \cos 90° & -\sin 90° & 0 & 0 \\ \sin 90° & \cos 90° & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

and

$$\begin{bmatrix} \cos -90° & -\sin -90° & 0 & 0 \\ \sin -90° & \cos -90° & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2.2 Determining the matrix to rotate about the y-axis

Using this model, let’s make a guess as to what the matrix would be to rotate about the y-axis. First notice that with the matrix $R_z$ the “z row” and “z column” are the “uninteresting” row and column (i.e. the row and column with just ones and zeros). Of course the fourth row and column are “uninteresting” (as the third row and column was in the 2D case) because of the three basic transformations we have considered (scale, rotate, and translate) only translate requires this fourth dimension.

So, as a guess for the matrix to rotate about the y-axis we have

$$R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & -\sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let’s verify if this matrix is correct as we did with $R_z$. Before we do, we must define a convention for what is considered a positive angle. Our convention for rotating a positive angle is: $+x \to +y \to +z$ and back again to $+x$. In other words, a positive rotation takes us from a positive $x$ value
to a positive $y$ value and from a positive $y$ value to a positive $z$ value and from a positive $z$ value to a positive $x$ value.

With this convention in place, let’s verify $R_y$.

\[
\begin{bmatrix}
\cos 90^\circ & 0 & -\sin 90^\circ & 0 \\
0 & 1 & 0 & 0 \\
\sin 90^\circ & 0 & \cos 90^\circ & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
= \begin{bmatrix}
-1 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Well, we are a bit off. To make the result correct, let’s change $-\sin 90^\circ$ in the first row and third column to $\sin 90^\circ$. Now, let’s continue verifying by rotating back again by $-90^\circ$.

\[
\begin{bmatrix}
\cos -90^\circ & 0 & \sin -90^\circ & 0 \\
0 & 1 & 0 & 0 \\
-\sin 90^\circ & 0 & \cos -90^\circ & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
-1
\end{bmatrix}
\]

Again we are off a bit. We can fix this easily by changing $\sin -90^\circ$ in the third row and first column to $-\sin -90^\circ$.

Now that we have correctly set up the matrix for $R_y$, what are the values for $\sin \theta$ and $\cos \theta$? We can obtain these with a bit of trigonometry. Notice from Figure 1 that,

\[\cos \theta = \frac{\text{side adjacent}}{\text{hypotenuse}} = \frac{c}{p}\]

and

\[\sin \theta = \frac{\text{side opposite}}{\text{hypotenuse}} = \frac{a}{p}\]

Finally, we have

\[
R_y(\theta) = \begin{bmatrix}
\frac{c}{p} & 0 & -\frac{a}{p} & 0 \\
0 & 1 & 0 & 0 \\
\frac{a}{p} & 0 & \frac{c}{p} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Let’s verify this result by multiplying the point $(a, b, c, 1)$ by the matrix $R_y$. We should obtain the point $(0, b, p, 1)$ (see Figure 2).

\[
\begin{bmatrix}
\frac{c}{p} & 0 & -\frac{a}{p} & 0 \\
0 & 1 & 0 & 0 \\
\frac{a}{p} & 0 & \frac{c}{p} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c \\
1
\end{bmatrix}
= \begin{bmatrix}
\frac{ac}{p} - \frac{ac}{p} \\
0 \\
\frac{a^2}{p} + \frac{c^2}{p} \\
1
\end{bmatrix}
= \begin{bmatrix}
0 \\
b \\
p \\
1
\end{bmatrix}
\]
3 Rotate about the $x$-axis

Now we must rotate about the $x$-axis so that the rotated center of interest $(0, b, p)$ is aligned with the $z$-axis (see Figure 2). Using the same reasoning and verifying as we did in subsection 2.2

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{b}{r} & \frac{-p}{r} & 0 \\ 0 & \frac{p}{r} & \frac{b}{r} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let’s verify this result by multiplying the point $(0, b, p, 1)$ by the matrix $R_x$. We should obtain the point $(0, 0, r, 1)$ (see Figure 3).

$$\begin{bmatrix} 0 \\ b \\ p \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{b}{r} & \frac{-p}{r} & \cos \alpha & \sin \alpha \\ \cos \alpha & \sin \alpha & -\sin \alpha & \cos \alpha \\ \sin \alpha & -\cos \alpha & \cos \alpha & \sin \alpha \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus the center of interest has now moved to align with the positive $z$-axis.

4 Reflect about the $y$-$z$ plane

Unfortunately as the viewer sees it, on his/her right is the $x$-axis. Since we normally prefer the positive $x$-axis on the viewer’s right, we negate the
Figure 3: After rotating about $x$-axis through an angle of $-\alpha$

$x$-values. The matrix to do this is

$$F = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

Thus, the complete world-to-eye transformation is:

$$W = FR_xR_yT$$

$$= \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{p}{r} & -\frac{b}{r} & 0 \\
0 & \frac{b}{r} & \frac{p}{r} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\frac{c}{p} & 0 & -\frac{a}{p} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & -e_x \\
0 & 1 & 0 & -e_y \\
0 & 0 & 1 & -e_z \\
0 & 0 & 0 & 1
\end{bmatrix}$$

Notice that the order is important since matrix multiplication is not generally commutative. However, matrix multiplication is associative so that we can perform the following first:

$$R_xR_y = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{p}{r} & -\frac{b}{r} & 0 \\
0 & \frac{b}{r} & \frac{p}{r} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\frac{c}{p} & 0 & -\frac{a}{p} & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
\frac{c}{p} & 0 & -\frac{a}{p} & 0 \\
-\frac{ab}{pr} & \frac{p}{r} & -\frac{bc}{pr} & 0 \\
\frac{a}{r} & \frac{b}{r} & \frac{c}{r} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$
Then we can perform

\[
F(R_x R_y) = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
\frac{c}{r} & 0 & -\frac{a}{r} & 0 \\
-\frac{a}{p} & \frac{c}{p} & 0 & 0 \\
0 & \frac{b}{r} & \frac{c}{r} & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-\frac{c}{p} & 0 & \frac{a}{p} & 0 \\
-\frac{a}{p} & \frac{c}{p} & 0 & 0 \\
\frac{a}{r} & \frac{b}{r} & \frac{c}{r} & 0 \\
\frac{a}{r} & \frac{b}{r} & \frac{c}{r} & 0 \\
\end{bmatrix}
\]

And finally we obtain

\[
(F(R_x R_y))T = \begin{bmatrix}
\frac{c}{p} & 0 & \frac{a}{p} & \frac{e_x}{p} \\
-\frac{a}{p} & \frac{c}{p} & 0 & \frac{e_y}{p} \\
\frac{a}{r} & \frac{b}{r} & \frac{c}{r} & \frac{e_z}{r} \\
\frac{a}{r} & \frac{b}{r} & \frac{c}{r} & \frac{e_z}{r} \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & -e_x \\
0 & 1 & 0 & -e_y \\
0 & 0 & 1 & -e_z \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{c}{p} & 0 & \frac{a}{p} & \frac{e_x}{p} - \frac{e_y}{p} \\
-\frac{a}{p} & \frac{c}{p} & 0 & \frac{e_y}{p} - \frac{e_z}{p} \\
\frac{a}{r} & \frac{b}{r} & \frac{c}{r} & -\frac{e_z}{r} \\
\frac{a}{r} & \frac{b}{r} & \frac{c}{r} & -\frac{e_z}{r} \\
\end{bmatrix}
\]